**European Journal of Applied Sciences – Vol. 13, No. 06**
**Publication Date:** December 27, 2025
**DOI:**10.14738/aivp.1306.19766.
Zahran, E. H. M. & Ertaş, A. G. (2025). New Variety Class of Solitons Arising From the (3+1)-Dimensional Evolution Equation. *European Journal of Applied Sciences, Vol - 13(06).* 279-289.

SERVICES FOR SCIENCE AND EDUCATION

# New Variety Class of Solitons Arising From the (3+1)-Dimensional Evolution Equation

**Emad H. M. Zahran**
Departments of Mathematical and Physical Engineering,
Benha University, Faculty of Engineering, Shubra, Egypt

**Ali Gökhan Ertaş**
Department of Informatics,
Kütahya Dumlupınar University, Kütahya, 43020, Türkiye

## ABSTRACT

**Our current work aimed constructing new variety soliton solutions to the completely integrable evolution (3+1)-dimensional nonlinear equation. The suggested model has strong relation with many equations especially with the Korteweg–De Vries (KDV) equation; it describes the real features in several branches of science as physics, fluid, engineering and technology. These soliton solutions of this model will be constructed for the first time using three distinct methods. The three employed methods are the generalized Kudryashov method (GKM), the extended direct algebraic method (EDAM) and (G'/G)-expansion method. The soliton solutions we obtained are novel compared to those previously reported by other authors using different methods.**

**Keywords:** The (3+1)-dimensional nonlinear evolution equation, generalized Kudryashov method, extended direct algebraic method, (G'/G)-expansion method, Soliton solutions.

## INTRODUCTION

In literature, the nonlinear lump waves, viewed as representative models for rogue wave dynamics in different scientific domains, have received notable attention.In this article we will study the Completely Integrable Evolution (3+1)-dimensional nonlinear equation that represents the real features in several branches of science like physics, fluid, engineering and technology; especially with the KDV equation. A variety of powerful techniques, as demonstrated in several important publications, have been developed and applied to study the soliton dynamics associated with this model; see, for example, In 2003, Geng began studying the algebraic and geometric solutions of certain multidimensional nonlinear evolution equations, and in subsequent years, he and Ma wrote the N-soliton solution for these nonlinear equations and the Wronskian form of the equation [1], [2].

Wazwaz made a significant contribution to science in 2009 with his work on multiple soliton and singular soliton solutions for (3+1) dimensional nonlinear evolution equations. Wazwaz also studied various types of multiple soliton solutions for these evolution equations in 2013 and 2014 and produced multiple soliton solutions [3], [5], [6].

Similarly, Yang, J.Y., and Ma, W.X., scientists who have worked on lump-type solutions of the Jimbo-Miwa equation in the same dimension, followed Zhaqilao's work on rogue waves and rational solutions for (3+1)-dimensional evolution equations [4], [7].

In 2017 and 2018, different scientists contributed to science with studies containing different solutions such as M-lump, mixed lump-kink, and lump-king for (3+1)-dimensional nonlinear evolution equations [8], [9], [10].

According to [6] the (3 + 1)-dimensional nonlinear evolution equation can be proposed as

$$3\Psi_{xz} - (2\Psi_t + \Psi_{xxx} - 2\Psi\Psi_x)_y + 2(\Psi_x \partial_x^{-1}\Psi_y)_x + 2(\Psi\partial_x^{-1}\Psi_{yy})_y + \Psi_{yz} = 0. \quad (1)$$

To remove the integral term, let us consider the potential

$$\Psi(x, y, z, t) = T_x(x, y, z, t). \quad (2)$$

Hence, Eq. (1) become

$$\begin{aligned} &3T_{xxz} - 2T_{xyt} - T_{xxxxy} + 2T_{xy}T_{xx} + 2T_xT_{xxy} \\ &+2T_{xxx}T_y + 2T_{xx}T_{xy} + 2T_{xy}T_{yy} + 2T_xT_{yyy} + T_{xyz} = 0. \end{aligned} \quad (3)$$

Let us consider the transformation $T(x, y, z, t) = T(\zeta)$, $\zeta = x + y + z - wt$, then Eq. (3) become

$$-T'''' + 6T'T''' + (2w + 4)T''' + 6T''^2 = 0. \quad (4)$$

To reduce this equation, let us take $T' = R$, then Eq. (4) become

$$-R''' + 6R R'' + (2w + 4)R'' + 6R'^2 = 0. \quad (5)$$

When the homogenous balance theory is applied for the above equation it implies that $M = 2$ The main objective of this study is to investigate (3+1)-dimensional nonlinear evolution equations by obtaining new soliton solutions using the three techniques mentioned above.The first one is the GKM [11, 12], the second one is EDAM [13-15], The (G′/G)-expansion technique constitutes the third approach. [16-18]. In Sections 2, 3, and 4 of our paper, the use of the GKT, EDAM, and (G'/G) algorithms, respectively, to generate soliton solutions for the proposed model is described in detail. In Section 5, the results are presented, and the paper is concluded.

## THE GKT ALGORITHM

To investigate the metholgy of this technique, let us consider the formalism of the nonlinear partial differential equation (NLPDE) in $R(x, y, z, \tau)$ and its partial derivatives which is:

$$\square(R, R_x, R_y, R_z, R_\tau, R_{xx}, R_{yy}, R_{\tau\tau}, R_{zz}, R_{xy}, R_{x\tau}, R_{y\tau}, R_{z\tau}, \dots) = 0. \quad (6)$$

That can be transformed to into ordinary differential equation in $R(x,y,z,t)$ and its total derivatives with the aid of the transformation $R(x,y,z,\tau)=R(\zeta), \zeta = x+y+z-w\tau$, in the form

$$\square(R, R', R'', R''', \ldots) = 0. \tag{7}$$

The GKM introduce the solution of Eq. (7) in the form:

$$R(\zeta)=\frac{\sum_{i=0}^{N} s_i q^i(\zeta)}{\sum_{j=0}^{M} g_j q^j(\zeta)}=\frac{s_0+s_1q(\zeta)+s_2q^2(\zeta)+\ldots}{g_0+g_1q(\zeta)+g_2q^2(\zeta)+\ldots}. \tag{8}$$

Where the parameters $s_i, (i=0,1,2,\ldots,N)$ and $g_j, (j=0,1,2,\ldots,M)$ will be defined subsequently in such a way that $s_N \neq 0 \,\&\, g_M \neq 0$ and thus the function $q(\zeta)$ is the solution of the second order nonlinear equation

$$\frac{dq(\zeta)}{d\zeta}=q^2(\zeta)-q(\zeta). \tag{9}$$

By integrating Eq. (9) we get

$$q(\zeta)=\frac{1}{1+Ke^{\zeta}}. \tag{10}$$

Where $K$ is the integration constsnty, to utilize the above schema?

The solution of Eq. (5) whose balance number is $M=2$, according to the GKT is:

$$R(\zeta)=\frac{s_0+s_1q(\zeta)+s_2q^2(\zeta)+s_3q^3(\zeta)}{g_0+g_1q(\zeta)+g_2q^2(\zeta)}. \tag{11}$$

By introducing $R, R', R'', R'''$ into Eq. (5), set the coefficients of various powers of $q^i$ =0 will give a equational system from which large several results will be detected, we will construct the solution of only one of them which is:

$$s_0=\frac{s_2g_0}{g_2}, s_1=\frac{s_2g_1}{g_2}, s_3=0, w=1. \tag{12}$$

**THIS RESULT CAN BE SIMPLIFIED TO BE**

$$s_0=s_1=-1, s_2=2, g_0=1, g_1=-2, g_2=4, =w=1, s_3=0. \tag{13}$$

According to these parameter values, the solution is:

$$R(\zeta)=\frac{-1-\left(\frac{1}{1+e^{\zeta}}\right)+2\left(\frac{1}{1+e^{\zeta}}\right)^{2}}{1-2\left(\frac{1}{1+e^{\zeta}}\right)+4\left(\frac{1}{1+e^{\zeta}}\right)^{2}}. \tag{14}$$

Hence;

$$T(\zeta)=\int R(\zeta)d\zeta.$$

$$T(\zeta)=0.5\zeta+0.25\ln[e^{2\zeta}-2e^{\zeta}+4]+\frac{3}{\sqrt{2e^{\zeta}-2}}. \tag{15}$$

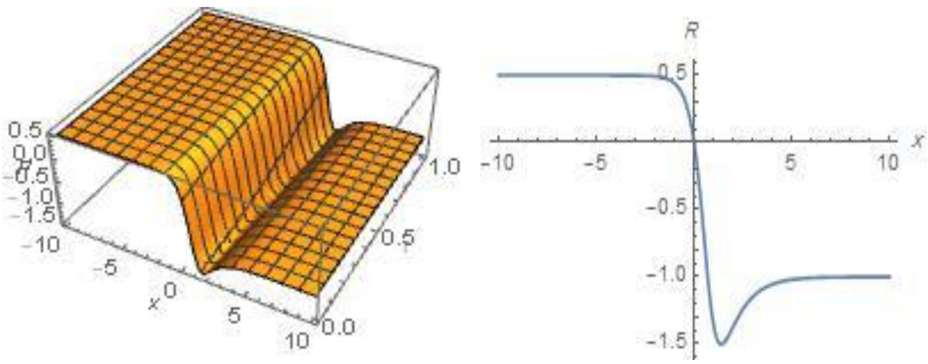


**Fig 1: The soliton solutions in two and three dimensions of $R(\zeta)$ Eq.(14) when:**

$s_0=s_1=-1, s_2=2, g_0=1, g_1=-2, g_2=4, =w=1, s_3=0, K=1.$

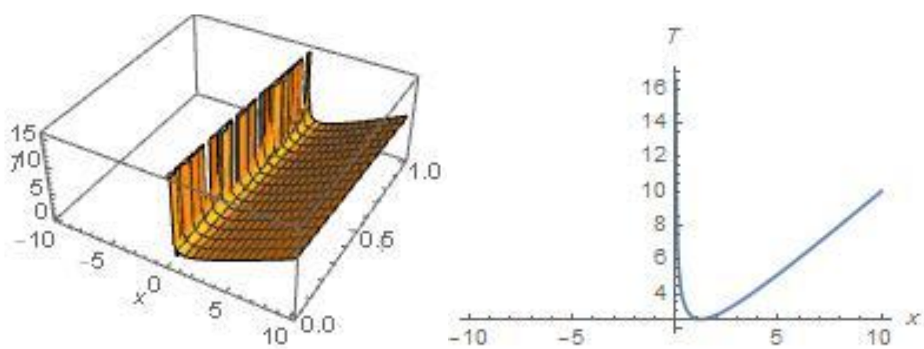


**Fig 2: The soliton solutions in two and three dimensions of $T(\zeta)$ Eq.(15) when:**

$s_0=s_1=-1, s_2=2, g_0=1, g_1=-2, g_2=4, =w=1, s_3=0, K=1.$

## THE EDAM ALGORITHM

The EDAM presents the solution of Eq. (7) as follows:

$$R(\zeta)=\sum_{i=0}^{M} b_i\ \varphi^{i}(\zeta),\ \ \varphi'^{2}=\alpha\varphi^{2}+\beta\varphi^{3}+\gamma\varphi^{4}. \tag{16}$$

Now, for the proposed model, the solution that satisfies the balance is:

$$R(\zeta) = b_0 + b_1\varphi + b_2\varphi^2. \tag{17}$$

Hence

$$R' = b_1\varphi' + 2b_2\varphi\varphi'. \tag{18}$$

$$R'' = b_1\varphi'' + 2b_2\varphi'^2 + 2b_2\varphi\varphi''. \tag{19}$$

$$R'''' = b_1\varphi'''' + 6b_2\varphi'' + 8b_2\varphi'\varphi''' + 2b_2\varphi\varphi''''. \tag{20}$$

$$\varphi'^2 = \alpha\varphi^2 + \beta\varphi^3 + \gamma\varphi^4 \tag{21}$$

$$\varphi'' = \alpha\,\varphi + 1.5\beta\,\varphi^2 + 2\gamma\,\varphi^3. \tag{22}$$

$$\varphi''' = \alpha\,\varphi' + 3\beta\,\varphi\,\varphi' + 6\gamma\,\varphi^2\varphi'. \tag{23}$$

$$\varphi'''' = \alpha\,\varphi'' + 3\beta\,\varphi\,\varphi'' + 3\beta\,\varphi\,\varphi'^2 + 12\gamma\,\varphi\,\varphi'^2 + 6\gamma\,\varphi^2\varphi''. \tag{24}$$

By setting the coefficients of the different forces in the above differential equations (17/24) to zero, 10 different results were obtained. One of these results is:

$$\alpha = -2, \beta = 0.7, \gamma = -0.1, b_0 = -1 - 0.3w\,, b_1 = 0.3, b_2 = -0.04. \tag{25}$$

The solution according to this result is:

$$R(\zeta) = \left(\frac{0.7 - 1.1\cos\sqrt{2}\zeta}{1.2 + 0.8\cos\sqrt{2}\zeta}\right) + i\left(\frac{2\sin\sqrt{2}\zeta\cos\sqrt{2}\zeta - 0.7\sin\sqrt{2}\zeta}{1.2 + 0.8\cos\sqrt{2}\zeta}\right). \tag{26}$$

Thus

$$\mathrm{Re}.(R(\zeta)) = \frac{0.7 - 1.1\cos\sqrt{2}\zeta}{1.2 + 0.8\cos\sqrt{2}\zeta}. \tag{27}$$

$$\mathrm{Re}T(\zeta) = \int \mathrm{Re}(R(\zeta))d\zeta.$$

$$\mathrm{Re}\,T(\zeta) = \frac{47\tan^{-1}(\frac{1}{\sqrt{5}}\tan\frac{\zeta}{\sqrt{2}})}{2^{2.5}\sqrt{5}} - \frac{11}{8}\zeta + C. \tag{28}$$

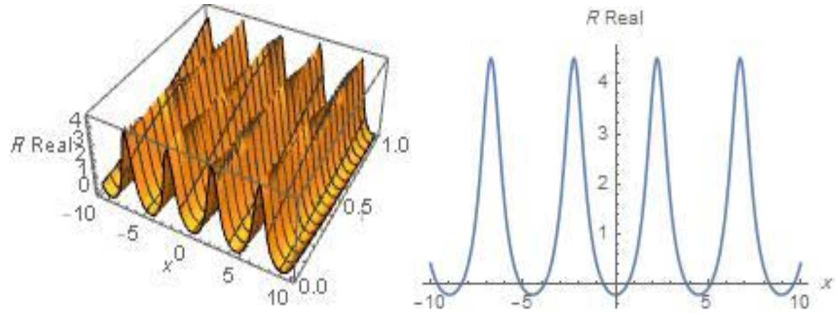


**Fig 3: The soliton solutions in two and three dimensions of $\operatorname{Re} R(\zeta)$ Eq.(27) when:**

$\alpha=-2, \beta=0.7, \gamma=-0.1, b_0=-1-0.3w, b_1=0.3, b_2=-0.04.$

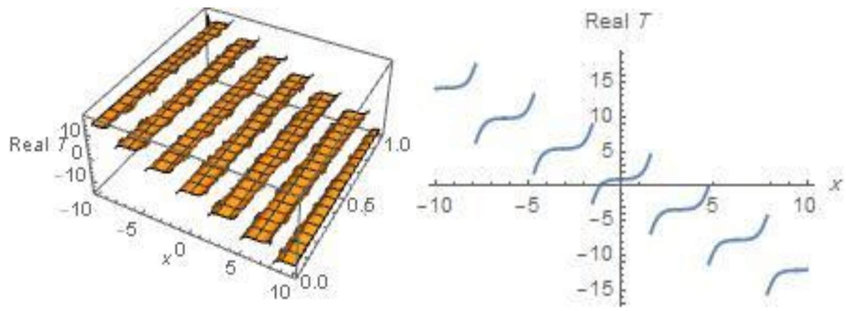


**Fig 4: The soliton solutions in two and three dimensions of $\operatorname{Re} T(\zeta)$ Eq.(28) when:**

$\alpha=-2, \beta=0.7, \gamma=-0.1, b_0=-1-0.3w, b_1=0.3, b_2=-0.04, C=1.$

And

$$\operatorname{Im}(R(\zeta))=\frac{2\sin\sqrt{2}\zeta\cos\sqrt{2}\zeta-0.7\sin\sqrt{2}\zeta}{1.2+0.8\cos\sqrt{2}\zeta}. \tag{29}$$

Hence

$$\operatorname{Im}T(\zeta)=\int \operatorname{Im}(R(\zeta))d\zeta.$$

$$\operatorname{Im}(T(\zeta))=\frac{3Ln[2\cos(\sqrt{2}\zeta)+3]-20\cos\sqrt{2}\zeta}{2^{3.5}}+C. \tag{30}$$

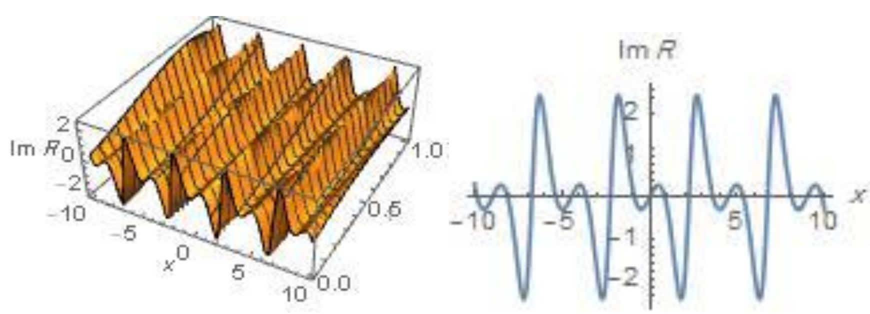


**Fig 5: The soliton solutions in two and three dimensions of $\operatorname{Im} R(\zeta)$ Eq.(29) when:**

$\alpha=-2, \beta=0.7, \gamma=-0.1, b_0=-1-0.3w, b_1=0.3, b_2=-0.04.$

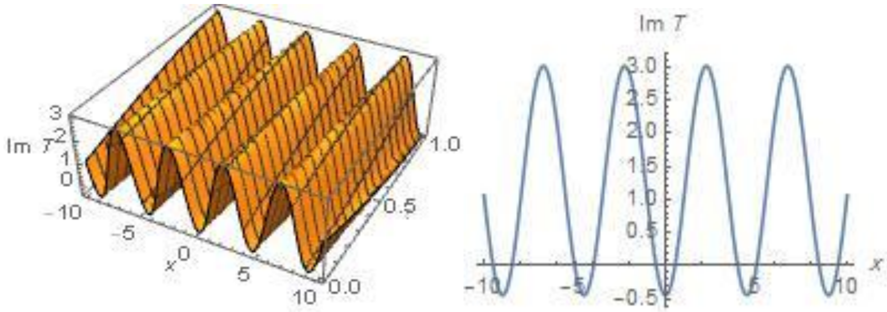


**Fig 6: The soliton solutions in two and three dimensions of $\mathrm{Im}T(\zeta)$ Eq.(30) when:**
$\alpha=-2, \beta=0.7, \gamma=-0.1, b_0=-1-0.3w, b_1=0.3, b_2=-0.04, C=1.$

## THE (G'/G) ALGORITHM

The (G'/G) algorithm presents the solution of Eq. (7) as:

$$R(\zeta)=A_0+\sum_{i=1}^{M}A_k\left[\frac{G'}{G}\right]^i, A_M\neq 0. \tag{31}$$

Where $G(\zeta)$ satisfies the auxiliary equation $G''+\mu G'+\lambda G=0$ for which these forms of solutions will be detected.

(I) If $\mu^2-4\lambda\succ 0$

$$\left(\frac{G'}{G}\right)=\frac{\sqrt{\mu^2-4\lambda}}{2}\left(\frac{s_1\sinh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta+s_2\cosh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}{s_1\cosh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta+s_2\sinh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}\right)-\frac{\mu}{2}. \tag{32}$$

(II) If $\mu^2-4\lambda\prec 0$ the solution is:

$$\left(\frac{G'}{G}\right)=\frac{\sqrt{\mu^2-4\lambda}}{2}\left(\frac{-s_1\sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta+s_2\cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}{s_1\cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta+s_2\sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}\right)-\frac{\mu}{2}. \tag{33}$$

(III) If $\mu^2-4\lambda=0$

$$\left(\frac{G'}{G}\right)=\left(\frac{s_2}{s_1+s_2\zeta}\right)-\frac{\mu}{2}. \tag{34}$$

The balance number MMM, calculated beforehand for the suggested model, is M=2. $s_1, s_2$ are constants, hence the solution according to this algorithm is:

$$R(\zeta)=A_0+A_1\left(\frac{G'}{G}\right)+A_2\left(\frac{G'}{G}\right)^2. \tag{35}$$

Hence

$$R' = -2A_2\left(\frac{G'}{G}\right)^3 - (A_1 + 2\mu A_2)\left(\frac{G'}{G}\right)^2 - (A_1\mu + 2A_2\lambda)\left(\frac{G'}{G}\right). \tag{36}$$

$$\begin{aligned} R'' = 6A_2\left(\frac{G'}{G}\right)^4 + (2A_1 + 10\mu A_2)\left(\frac{G'}{G}\right)^3 + (3A_1\mu + 8A_2\lambda + 2\mu A_2)\left(\frac{G'}{G}\right)^2 \\ + (2A_1\lambda + A_1\mu^2 + 6A_2\lambda\mu)\left(\frac{G'}{G}\right) + A_1\lambda\mu + 2A_2\lambda^2. \end{aligned} \tag{37}$$

$$\begin{aligned} R'''' = 120A_2\left(\frac{G'}{G}\right)^6 + (24A_1 + 336\mu A_2)\left(\frac{G'}{G}\right)^5 \\ + (60A_1\mu + 306A_2\mu^2 + 240A_2\lambda + 12\mu A_2)\left(\frac{G'}{G}\right)^4 \\ + (40A_1\lambda + 50A_1\mu^2 + 380A_2\lambda\mu + 20A_2\mu^2 + 90A_2\mu^3 + 60A_1A_2\mu)\left(\frac{G'}{G}\right)^3 \\ + (60A_1\lambda\mu + 136A_2\lambda^2 + 140\lambda\mu^2A_2 + 15A_1\mu^3 + 16A_2\lambda\mu + 8A_1A_2\mu^2)\left(\frac{G'}{G}\right)^2 \\ + (16A_1\lambda^2 + 22\lambda\mu^2A_1 + 12\lambda\mu^2A_2 + 60\mu\lambda^2A_2 + A_1\mu^4 + 6A_2\lambda\mu^3 + 60A_1A_2\lambda\mu)\left(\frac{G'}{G}\right)^2 \\ + 8\mu\lambda^2A_1 + 4\mu\lambda^2A_2 + 6\mu^2\lambda^2A_2 + 16\lambda^3A_2 + \lambda\mu^3A_1. \end{aligned} \tag{38}$$

When R and its partial derivatives are emerged inside Eq. (5), by equating the coefficients of various powers $(\frac{G'}{G})^i$ to zero, a system of equations with two different results was obtained. which we will construct the solution of one of them which is:

$$\lambda = 0, \mu = \frac{A_1}{2}, A_2 = 2, w = \frac{1}{2}(-4 + \mu^2 - 6A_0). \tag{39}$$

This result can be simplified to be

$$\lambda = 0,, \mu = 0.5, A_0 = A_1 = A_1 = 1, w = -5. \tag{40}$$

The solution according to this result is:

$$R(\zeta) = 1 + \left(\frac{G'}{G}\right) + \left(\frac{G'}{G}\right)^2.$$

Where

$$(\frac{G'}{G}) = \left(\frac{0.25\sinh 0.25\zeta + 0.5\cosh 0.25\zeta}{\cosh 0.25\zeta + 2\sinh 0.25\zeta}\right) - 0.25.$$

Thus

$$R(\zeta)=1+\left(\frac{0.25\cosh 0.25\zeta-0.25\sinh 0.25\zeta}{\cosh 0.25\zeta+2\sinh 0.25\zeta}\right)+\left(\frac{0.25\cosh 0.25\zeta-0.25\sinh 0.25\zeta}{\cosh 0.25\zeta+2\sinh 0.25\zeta}\right)^2. \quad (41)$$

And

$$T(\zeta)=\int R(\zeta)\,d\zeta.$$

$$T(\zeta)=\frac{3\zeta}{4}+\ln[\cosh\frac{\zeta}{4}+2\sinh\frac{\zeta}{4}] + \frac{Ln[\tanh\frac{\zeta}{4}+1]-Ln[2\tanh\frac{\zeta}{4}+1]}{2}-\frac{3}{16\tanh\frac{\zeta}{4}+8}+C. \quad (42)$$

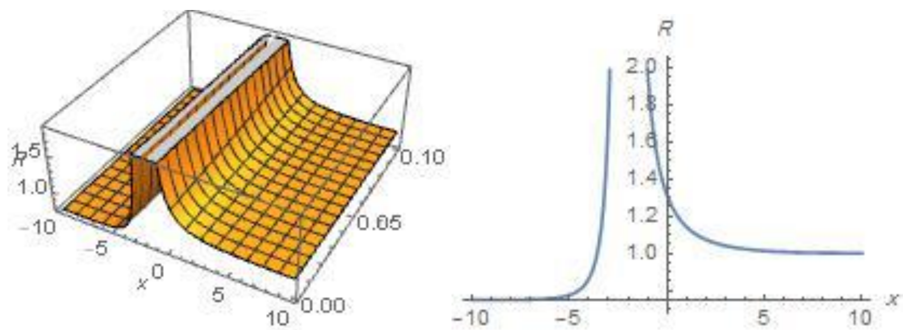

**Fig 7: The soliton solutions in two and three dimensions of $R(\zeta)$ Eq.(41) when:**

$\lambda=0,,\mu=0.5,A_0=A_1=A_1=1,s_1=1,s_2=2,w=-5.$

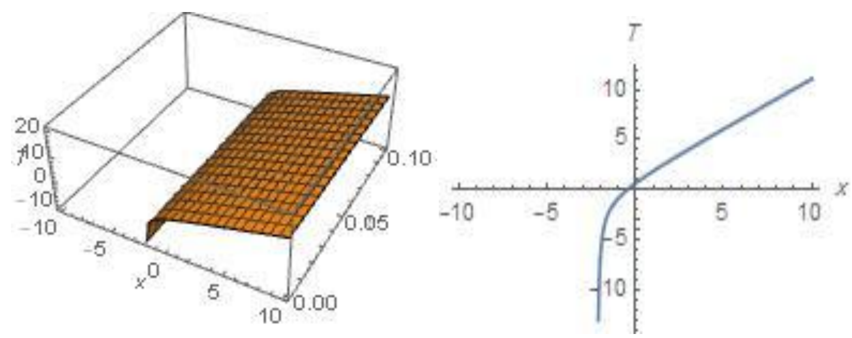

**Fig 8: The soliton solutions in two and three dimensions of $T(\zeta)$ Eq.(42) when:**

$\lambda=0,,\mu=0.5,A_0=A_1=A_1=1,s_1=1,s_2=2,w=-5,C=1.$

## CONCLUSION

In our paper, we obtained very important results for (3+1)-dimensional nonlinear equations using three different expansion methods to obtain soliton solutions. These three methods were applied in the same vein and in parallel, and through these methods, new types of soliton solutions were presented, such as parabolic and hyperbolic function soliton solutions and combinations of bright and dark soliton solutions. Additionally, singular soliton solutions and other rational soliton solutions were also presented. The 2D and 3D behaviors of these solutions represent a new vision of the soliton emerging from this model. The novelty of the results obtained in our study will be more understandable when compared with studies documented in [6,10].

## Data Availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

## Conflict of Interest

The authors declare that they have no conflict of interest.

## References


[1] Geng X. (2003); Algebraic-geometrical solutions of some multidimensional nonlinear evolution equations; J. Phys. A: Math. Gen. 36, 2289–2303

[2] Geng X. and Ma, Y. (2007); N-soliton solution and its Wronskian form of a (3+1)-dimensional nonlinear evolution equation, Phys. Lett. A 369(4), 285– 289

[3] Wazwaz A. M. (2009); A (3+1)-dimensional nonlinear evolution equation with multiple soliton solutions and multiple singular soliton solutions, Appl. Math. Comput. 215(4), 1548–1552

[4] Zhaqilao (2013); Rogue waves and rational solutions of a (3+1)-dimensional nonlinear evolution equation, Physics Letters A 37, 3021-3026

[5] Wazwaz A. M., A. (2013); variety of distinct kinds of multiple soliton solutions for a (3+1)-dimensional nonlinear evolution equations, Mathematical Methods in Applied Sciences 36, 2013, 349–357

[6] Wazwaz A. M., A. (2014); New (3+1)-dimensional nonlinear evolution equation: multiple soliton solutions; Cent. Eur. J. Eng.; 4(4); 352-356

[7] Yang, J.Y., Ma, W.X. (2017); Abundant lump-type solutions of the jimbo-miwa equation in (3+1)-dimensions, Comput. Math. Appl. 73 (2); 220–225.

[8] Zhang, J.B., Ma, W.X. (2017); Mixed lump-kink solutions to the bkp equation, Comput. Math. Appl. 74 (3); 591–596.

[9] Zhao, H.Q., Ma, W.X. (2017); Mixed lump-kink solutions to the kp equation, Comput. Math. Appl. 74 (6); 1399–1405.

[10] Zhang,Y., Liu, Y., Tang, X.(2018); M-lump solutions to a (3 + 1)-dimensional nonlinear evolution equation

[11] Loubna Ouahid, Saud Owyed , M.A. Abdou , Nawal A. Alshehri , S.K. Elagan, 2021; New optical soliton solutions via generalized Kudryashov's scheme for Ginzburg–Landau equation in fractal order; Alexandria Engineering Journal; 60 (6); pages 5495-5510.

[12] Dipankar Kumar , Aly R. Seadawy, Atish Kumar Joardar, (2018); Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology; Chinese Journal of Physics; 56 (1) pages 75-85.

[13] Zahran, E., Bekir, A. (2022), Enormous soliton solutions to a (2+1)-dimensional Heisenberg ferromagnetic spin chain equation, Chinese Journal of Physics 77 1236–1252

[14] Zahran, E., Hijaz, A., Askar, S., Botmart, T., Shehata, M. (2022); Dark-soliton behaviors arising from a coupled nonlinear Schrödinger system, Results in Physics 36: 105459

[15] Zahran, E., Bekir, A. (2023); New diverse soliton solutions for the coupled Konno-Oono equations; Optical and Quantum Electronics; 55:112

[16] Zahran, E., Bekir, A. (2022); Unexpected configurations for the optical solitons propagation in lossy fiber system with dispersion terms effect; Wiley; Math Meth Appl Sci. 2022;1–15.; DOI: 10.1002/mma.8738

[17] Zahran, E., Bekir, A., Ibrahim, R. New impressive analytical optical soliton solutions to the Schrödinger-Poisson dynamical system against its numerical solutions. Opt. Quant. Electron. 55, 212 (2023)

[18] Bekir, A. Zahran, E., Shehata, M., Comparison between the new exact and numerical solutions of the Mikhailov-Novikov-Wang equation, Numerical Method of Partial Differential Equation, 40, e22775 (2024)